

Canonical structure of higher derivative theories

Eran Avraham, Ram Brustein

Department of Physics, Ben-Gurion University,

Beer-Sheva 84105, Israel

E-mail: eranav@bgu.ac.il, ramyb@bgu.ac.il

Abstract

The canonical structure of theories whose Lagrangian contains higher powers of time derivatives is often obscured by the nonlinear relationship between the velocities and momenta. We use the Dirac formalism and define a generalized Legendre transform to overcome some of the difficulties associated with inverting the relation between velocities and momenta. We are then able to define a standard single valued symplectic structure on phase space and a compatible single valued Hamiltonian. We demonstrate the application of our formalism in several examples.

1 INTRODUCTION

Theories containing higher derivatives (HD) appear in physics in several contexts. The simple cases when the Lagrangian is quadratic in velocities are only approximate, valid at small velocities. These simple Lagrangians can be viewed as the leading order expansion of more general effective Lagrangians. In field theories, in addition to higher time derivatives, one also encounters higher space derivatives. In relativistic field theories, the fields are a function of spacetime and the expansion is in spacetime derivatives.

Not all theories containing higher derivatives are sensible. If their equations of motion contain terms with more than two time derivatives, then they possess runaway solutions, signaling an instability at the shortest time scales. We are not interested in such theories. So, we will limit ourselves to theories whose equations of motion do not exhibit such instabilities. These theories have a Lagrangian that is typically a polynomial in velocities and does not depend explicitly on time derivatives of the velocities.

The canonical formulation of HD theories is an important step toward consistently quantizing them. The Ostrogradsky method [1] (See also [2]) enables one to canonically formulate a wide class of HD theories and to derive a Hamiltonian for them. We discuss a class of theories to which the Ostrogradsky method cannot be applied. As mentioned above, our focus in the paper is on theories containing high powers of the velocities. In this case there are too many solutions to the inversion equation connecting velocities and momenta, leading to a multivalued Hamiltonian. This situation cannot be handled in the Ostrogradsky method because (i) The Lagrangian does not contain new degrees of freedom and (ii) The inversion of the velocities in

terms of momenta is multivalued.

Over the years, it became clear that when attempting this task one encounters various difficulties. The issue that we would like to understand is whether the difficulties are fundamental and result from some basic obstruction to applying the canonical formalism to such theories. If this conclusion is verified, it might indicate that the canonical formalism itself is limited in some ways. Alternatively, the difficulty could be technical, specific to some subclasses of such theories and originate from their added complexity. The latter possibility indicates that we can apply the canonical formalism in general and develop approximation schemes when its application becomes technically difficult. Our results support this case.

We address, in particular, the problem of finding a Hamiltonian for the class of HD theories of interest. The method that we have found to be suitable for this task is the Dirac formalism. In this formalism, one extends phase space by adding new variables and then imposes constraints to remove the additional variables. This method allows us to define good phase space coordinates in which one can solve the constraints and reduce the number of variables to the original number of degrees of freedom. The end result is a single valued Hamiltonian which is compatible with the symplectic structure on phase space.

We find that the obstructions are technical and not fundamental. They are equivalent to the difficulty of solving nonlinear equations in standard theories with complicated interactions that involve only the coordinates.

One of the main motivations for the current investigations is to understand the canonical structure of theories of gravity. There, the expansion of

the effective Lagrangian is in terms of curvature invariants. In this context we wish to understand the canonical structure of a class of HD theories of gravity, such as $f(R)$ gravity and Lovelock gravity [3]. Such theories pose difficulties for implementing the Hamiltonian formalism. The high powers of velocities in the Lagrangian lead to a complicated algebraic equation connecting the canonical momentum and the velocity $P = \frac{\partial L}{\partial \dot{x}}$. This in turn leads to a multivalued Hamiltonian in terms of P . This issue was addressed several times in the literature [4,5,6,7,8,9,10] using different approaches.

The plan of the paper is as follows. In Sec. 2 we present our formalism for the case of a single variable. We explain how to use the Dirac formalism and how to define the generalized Legendre transform. We then discuss several examples. In Sec. 3 we extend the formalism for the case of several variables. We conclude by presenting a summary of results and conclusions.

2 GENERALIZED LEGENDRE TRANSFORM FOR A SINGLE VARIABLE

In this section we describe how to use the Dirac formalism for HD theories and how to choose an appropriate set of coordinates in phase space.

2.1 Formalism

Let us consider the Lagrangian $L(x, \dot{x})$ of a single dynamical variable. Rather than treating \dot{x} as the time derivative of $x(t)$, we wish to treat \dot{x} as an independent variable. The new independent variable will be labeled by Q .

To this end we add a Lagrange multiplier λ that imposes the equality of Q and \dot{x} at all times. Then the new Lagrangian is given by

$$\tilde{L} = L(x, Q) + \lambda(\dot{x} - Q). \quad (2.1)$$

Because the resulting Lagrangian is singular: the momentum of Q and λ vanishes, we turn to the Dirac formalism [11]. Performing a Legendre transform we obtain the canonical Hamiltonian,

$$H_C = \lambda Q - L(x, Q) \quad (2.2)$$

and three primary constraints,

$$\phi_1 = P_Q, \quad \phi_3 = P_\lambda, \quad \phi_4 = P_x - \lambda. \quad (2.3)$$

The constraints ϕ_3, ϕ_4 are not dynamical and can be harmlessly substituted into the modified Hamiltonian (see [12]),

$$\begin{aligned} H_1 &= H_C + u_1 \phi_1 \\ &= P_x Q - L(x, Q) + u_1 P_Q. \end{aligned} \quad (2.4)$$

Demanding consistency from the constraint $\{\phi_1, H_1\} = -\partial_Q H_1$ we find a secondary one

$$\phi_2 = P_x - \partial_Q L. \quad (2.5)$$

Including this constraint in the Hamiltonian results in the total Hamiltonian,

$$H_T = P_x Q - L(x, Q) + u_1 P_Q + u_2 (P_x - \partial_Q L). \quad (2.6)$$

On shell, we may set the constraints to zero, provided that we use the Dirac brackets rather than the Poisson brackets [11]. The Poisson brackets

between the two second class constraints are given by

$$\{\phi_1, \phi_2\} = -\partial_Q \phi_2 = \partial_Q^2 L(x, Q), \quad (2.7)$$

so the Dirac matrix and its inverse are given by

$$M_{ij} = \partial_Q^2 L(x, Q) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_{ij}^{-1} = -\frac{1}{\partial_Q^2 L(x, Q)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.8)$$

Then the Dirac brackets for any pair of physical quantities A, B , is given by

$$\{A, B\}_D = \{A, B\} + \frac{1}{\partial_Q^2 L(x, Q)} \left[\{A, \phi_1\} \{\phi_2, B\} - \{A, \phi_2\} \{\phi_1, B\} \right].$$

We can now calculate the final Hamiltonian

$$H_F = Q \partial_Q L(x, Q) - L(x, Q). \quad (2.9)$$

However, this Hamiltonian is valid with the caveat that the Dirac bracket between phase space variables is not given by the standard expression $\{x, Q\}_D = 1$, rather it is given by the inverse of the Hessian,

$$\{x, Q\}_D = (\partial_Q^2 L(x, Q))^{-1}. \quad (2.10)$$

To proceed we need to change coordinates in phase space: we need to find some new functions, say $f(x, Q)$ and $g(x, Q)$ such that $\{f, g\}_D = 1$. Noting that ϕ_1 acts like ∂_Q and ϕ_2 like ∂_x , we can write the condition explicitly:

$$\partial_x f \partial_Q g - \partial_Q f \partial_x g = \partial_Q^2 L(x, Q). \quad (2.11)$$

Once two such functions are found, we need to invert the relations between (x, Q) and (f, g) and reexpress the final Hamiltonian in terms of new canonical variables.

When a pair of functions (f, g) that satisfies condition (2.11) is found, it is possible to use the standard Poisson brackets with respect to new basis (f, g) rather than the Dirac brackets. This follows from the chain rule property that both the Poisson and Dirac brackets posses:

$$\begin{aligned}\dot{f}(x, Q) &= \left\{ f(x, Q), H(f(x, Q), g(x, Q)) \right\}_D \\ &= \{f, f\}_D \frac{\partial H}{\partial f} + \{f, g\}_D \frac{\partial H}{\partial g} = \frac{\partial H}{\partial g},\end{aligned}\quad (2.12)$$

where the last equality is due to condition (2.11) which guarantees that $\{f, g\}_D = 1$. The final result is therefore

$$\dot{f}(x, Q) = \{f, H\}_{(f,g)}. \quad (2.13)$$

The standard definition of canonical conjugates $f = x$, $g = \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial Q}$, is a particular case of a more general relation (2.11). This freedom to choose a different set coordinates in phase space can be used to bypass some of the difficulties one may encounter using the standard Legendre transform.

2.2 A simple example

Consider the simple HD Lagrangian

$$L = \frac{1}{4}\dot{x}^4 - \frac{1}{2}k\dot{x}^2 - \frac{1}{2}\omega x^2, \quad (2.14)$$

where $k, \omega > 0$ are constants. A similar Lagrangian (with $\omega = 0$) was discussed recently in [4,5,6,7,8].

If we perform the standard Legendre transform we encounter the cubic equation for the momentum P_x in terms of the velocity \dot{x} ,

$$P_x = \dot{x}^3 - k\dot{x}. \quad (2.15)$$

The inversion of \dot{x} in terms of P_x leads to a multivalued Hamiltonian, with cusps at minima (see Fig. 1). The lowest energy solution of system “spontaneously breaks time translation” [2], because at the cusps the velocity is nonvanishing $\dot{x} = \pm\sqrt{k/3}$.

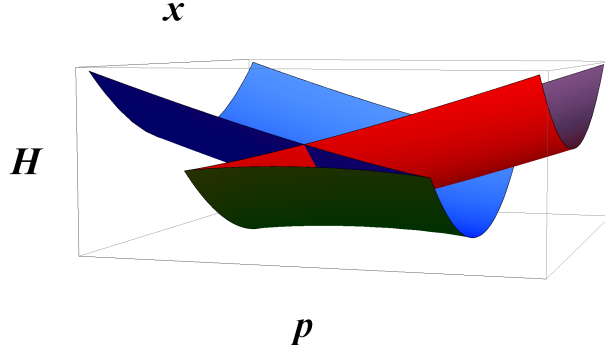


Figure 1: Multivalued Hamiltonian

Applying the generalized Legendre transform procedure that we have described in the previous subsection proceeds as follows: We change variables to (x, Q)

$$L \rightarrow \tilde{L} = \frac{1}{4}Q^4 - \frac{1}{2}kQ^2 - \frac{1}{2}\omega x^2 + \lambda(\dot{x} - Q). \quad (2.16)$$

The corresponding canonical Hamiltonian is

$$H_C = \frac{3}{4}Q^4 - \frac{1}{2}kQ^2 + \frac{1}{2}\omega x^2 \quad (2.17)$$

and the two second class constraints are

$$\phi_1 = P_Q, \quad \phi_2 = P_x - Q^3 + kQ. \quad (2.18)$$

We turn to find f and g , satisfying the condition

$$\partial_x f \partial_Q g - \partial_Q f \partial_x g = \partial_Q^2 L(x, Q)$$

$$= 3Q^2 - k \quad (2.19)$$

Choosing $g = Q$ determines f up to some function $\psi(Q)$ which we set to zero

$$f = x(3Q^2 - k). \quad (2.20)$$

Now the inversion is simple

$$Q = g, \quad (2.21)$$

$$x = \frac{f}{3Q^2 - k} = \frac{f}{3g^2 - k}. \quad (2.22)$$

Substituting x and Q in terms of f and g , we find the final Hamiltonian which is single valued and now has two standard canonical conjugate variables satisfying the standard Poisson brackets, $\{f, g\} = 1$,

$$H(f, g) = \frac{\omega}{2(3g^2 - k)^2} f^2 + \frac{3}{4} g^4 - \frac{1}{2} k g^2. \quad (2.23)$$

The resulting Hamiltonian effectively describes a particle in a quartic potential with a divergent mass term.

The Hamilton equations are given by

$$\dot{g} = -\frac{\partial H}{\partial f} = -\frac{\omega}{(3g^2 - k)^2} f, \quad (2.24)$$

$$\dot{f} = \frac{\partial H}{\partial g} = 3g^3 - kg - 6\omega \frac{gf^2}{(3g^2 - k)^3}. \quad (2.25)$$

Which, using Eqs. (2.21–2.22), reproduce the same equations of motion derived from the starting Lagrangian (2.14),

$$(3\dot{x}^2 - k)\ddot{x} = -\omega x. \quad (2.26)$$

The Hamiltonian of Eq. (2.23) is plotted in Fig. 2. It has two minima at $f = 0$, $g = \pm\sqrt{k/3}$. So the expected nonvanishing values of the velocities

are now a result of minimizing the Hamiltonian rather than the branched nature of phase space.

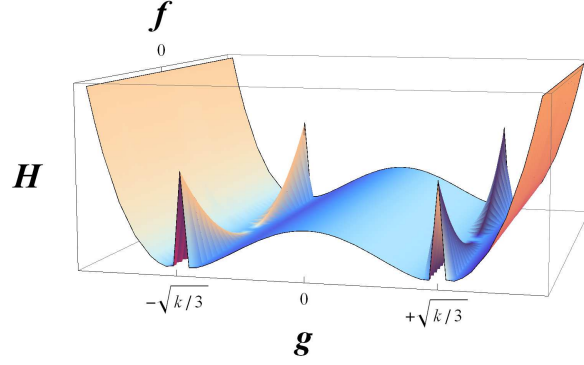


Figure 2: Single valued Hamiltonian

2.3 Another example: Minisuperspace Gauss-Bonnet

As explained in the Introduction, HD theories appear in the context of generalized theories of gravity. There, higher curvature terms added to the action corresponds to high powers of “velocity.” In the following we will derived the Hamiltonian for a simple Gauss-Bonnet cosmological model. We do not include here the lapse function and the associated Hamiltonian constraint. This is discussed, for example, in [14],[15].

The Gauss-Bonnet Lagrangian, is given by

$$L_{GB} = \sqrt{-g}[R + \gamma(R^2 - 4R_{\alpha\beta}^2 + R_{\alpha\beta\gamma\delta}^2)] \quad (2.27)$$

Assuming a metric of the form

$$ds^2 = -dt^2 + a(t)^2 dx_i^2 \quad (2.28)$$

in $D = 5$ with zero spatial curvature, we use (2.28) in (2.27) and integrate by parts (see [14],[15]) to find the Gauss-Bonnet minisuperspace Lagrangian,

$$L_{GB} = \frac{1}{2}a^2\dot{a}^2 + \frac{1}{4}\alpha\dot{a}^4. \quad (2.29)$$

As in the previous example, standard Legendre transform will lead to a multivalued Hamiltonian. Applying our method, we first need to change variables $a \rightarrow e^x$, changing (2.29) to

$$\tilde{L}_{GB} = e^{4x}[\frac{1}{2}\dot{x}^2 + \frac{1}{4}\alpha\dot{x}^4]. \quad (2.30)$$

Adding the Lagrange multiplier and switching to the (x, Q) coordinates we find the corresponding canonical Hamiltonian,

$$H_C = e^{4x}[\frac{1}{2}Q^2 + \frac{3}{4}\alpha Q^4]. \quad (2.31)$$

Solving the generalized Legendre equation (2.11) requires that

$$\partial_x f \partial_Q g - \partial_Q f \partial_x g = e^{4x}[1 + 3\alpha Q^2]. \quad (2.32)$$

We choose $g = Q$, and find that

$$f = \frac{1}{4}e^{4x}[1 + 3\alpha Q^2]. \quad (2.33)$$

Inverting (x, Q) in terms of (f, g) and substituting into Eq. (2.31) we obtain the final Hamiltonian

$$H(f, g) = fg^2\left[1 + \frac{1}{3\alpha g^2 + 1}\right]. \quad (2.34)$$

If $\alpha < 0$ the minimum of the Hamiltonian lies in $g = \pm\sqrt{\frac{2}{3|\alpha|}}$ corresponding to $a = e^{\pm\sqrt{\frac{2}{3|\alpha|}}t}$, that is, an expanding or contracting universe.

3 THE GENERALIZED LEGENDRE TRANSFORM FOR SEVERAL VARIABLES

Next we generalize our formalism for the case of several variables.

3.1 Formalism

Our starting point is now a Lagrangian which is a functional of several variables $L(x_i, \dot{x}_i)$. As in the previous section, we add the Lagrange multipliers

$$L(x_i, \dot{x}_i) \rightarrow L(x_i, Q_i) + \sum_i \lambda_i (\dot{x}_i - Q_i). \quad (3.1)$$

The corresponding canonical Hamiltonian is given by

$$H_C = \sum_i P_{x_i} Q_i - L(x_i, Q_i). \quad (3.2)$$

The primary and secondary constraints are the following,

$$\phi_i^1 = P_{Q_i}, \quad \phi_i^2 = P_{x_i} - \frac{\partial L}{\partial Q_i} \quad (3.3)$$

and their Poisson brackets are the following,

$$\{\phi_i^1, \phi_j^1\} = 0, \quad (3.4)$$

$$\{\phi_i^1, \phi_j^2\} = \frac{\partial^2 L}{\partial Q_i \partial Q_j} \equiv W_{ij}, \quad (3.5)$$

$$\{\phi_i^2, \phi_j^2\} = \frac{\partial^2 L}{\partial x_i \partial Q_j} - \frac{\partial^2 L}{\partial x_j \partial Q_i} \equiv B_{ij}. \quad (3.6)$$

The Dirac matrix and its inverse can be expressed in terms of the matrices W_{ij} and B_{ij} ,

$$M_{IJ} = \left(\begin{array}{c|c} 0 & W_{ij}^{-1} \\ \hline -W_{ij}^{-1} & B_{ij} \end{array} \right), \quad M_{IJ}^{-1} = \left(\begin{array}{c|c} W_{ik}^{-1} B_{kl} W_{lj}^{-1} & -W_{ij}^{-1} \\ \hline W_{ij}^{-1} & 0 \end{array} \right) \quad (3.7)$$

Thus the Dirac Brackets are given by

$$\begin{aligned} \{f(x, Q), g(x, Q)\}_D &= \{f, g\} - \sum_{IJ} \{f, \phi_I\} M_{IJ}^{-1} \{\phi_J, g\} \\ &= 0 - \sum_{ij} \left(\dots, \frac{\partial f}{\partial Q_i}, \dots, \frac{\partial f}{\partial x_i}, \dots \right) \left(\begin{array}{c|c} W_{ik}^{-1} B_{kl} W_{lj}^{-1} & -W_{ij}^{-1} \\ \hline W_{ij}^{-1} & 0 \end{array} \right) \begin{pmatrix} \vdots \\ -\frac{\partial g}{\partial Q_j} \\ \vdots \\ -\frac{\partial g}{\partial x_j} \\ \vdots \end{pmatrix} \\ &= \sum_{ij} \left[W_{ij}^{-1} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial Q_j} - \frac{\partial f}{\partial Q_i} \frac{\partial g}{\partial x_j} \right) + W_{ik}^{-1} B_{kl} W_{lj}^{-1} \frac{\partial f}{\partial Q_i} \frac{\partial g}{\partial Q_j} \right]. \end{aligned} \quad (3.8)$$

We can now write the generalized condition for the canonical pairs

$$\begin{aligned} \{f_a, g_b\}_D &= \sum_{ij} \left[W_{ij}^{-1} A_{ij} + T_{ij} C_{ij} \right] \\ &= \text{Tr}(W^{-1} A^T + T C^T) \\ &= \delta_{ab}. \end{aligned} \quad (3.9)$$

In the derivation of Eq.(3.9) we have used the identity $\sum_{ij} E_{ij} F_{ij} = \text{Tr}(E F^T)$ and introduced the following notations

$$A_{ij} \equiv \left(\frac{\partial f_a}{\partial x_i} \frac{\partial g_b}{\partial Q_j} - \frac{\partial f_a}{\partial Q_i} \frac{\partial g_b}{\partial x_j} \right), \quad T_{ij} \equiv W_{ik}^{-1} B_{kl} W_{lj}^{-1}, \quad C_{ij} \equiv \frac{\partial f_a}{\partial Q_i} \frac{\partial g_b}{\partial Q_j}. \quad (3.10)$$

Let us verify that our formalism can accommodate the standard choice of $f_a = x_a$ and $g_a = \frac{\partial L}{\partial Q_a}$. In this case A^T and C are given by

$$A_{ij}^T = \begin{pmatrix} 0 & \cdots & \frac{\partial L}{\partial Q_a \partial Q_1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & \frac{\partial L}{\partial Q_a \partial Q_i} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} 0 & \cdots & W_{1a} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & W_{ia} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \end{pmatrix}, \quad C_{ij} = 0. \quad (3.11)$$

So, we find

$$\{f_a, g_a\}_D = \text{Tr}(W_{ik}^{-1} A_{kj}^T) = \text{Tr} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ & \ddots & & & \\ 0 & \cdots & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} = 1. \quad (3.12)$$

One can also check that $\{x_a, x_b\}_D, \{x_a, \frac{\partial L}{\partial Q_b}\}_D, \{\frac{\partial L}{\partial Q_a}, \frac{\partial L}{\partial Q_b}\}_D = 0$ for $a \neq b$ as expected.

3.2 The case of $B_{ij} = 0$

If the coupling between the velocities and the coordinates vanishes, then the matrix $B_{ij} = \frac{\partial^2 L}{\partial x_i \partial Q_j} - \frac{\partial^2 L}{\partial x_j \partial Q_i}$ vanishes and condition (3.9) reduces to

$$\{f_a, g_b\}_D = \text{Tr}(W^{-1} A^T) = \delta_{ab}. \quad (3.13)$$

In this case, the added complexity is due only to the higher powers of the velocity (say a term of the form \dot{x}^4). It is useful to choose $g_a = Q_a$ which determines the form of A_{kj}^T ,

$$A_{kj}^T = \begin{pmatrix} 0 & \cdots & \frac{\partial f_a}{\partial x_1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & \frac{\partial f_a}{\partial x_i} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \end{pmatrix}. \quad (3.14)$$

Substituting into Eq. (3.13) we find the condition for f_a is the following

$$\frac{\partial f_a}{\partial x_i} = W_{ia}, \quad (3.15)$$

which has a solution only if W_{ia} is an exact differential.

In order to complete the analysis we prove that all other brackets vanish, i.e

$$\{Q_a, Q_b\}_D, \{g_a, g_b\}_D, \{Q_a, g_b\}_D, \{Q_b, g_a\}_D = 0. \quad (3.16)$$

The first condition comes from the definition of A_{ij} . The second from the requirement that $B_{ij} = 0$. For the third and fourth conditions let us write explicitly

$$A_{kj}^T(f_a, Q_b) = \begin{pmatrix} 0 & \dots & \frac{\partial f_a}{\partial x_1} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \frac{\partial f_a}{\partial x_i} & \dots & 0 \\ \vdots & & \vdots & & \vdots \end{pmatrix}. \quad (3.17)$$

Now, the “ b ” column is occupied by $\frac{\partial f_a}{\partial x_i}$ which equals by construction to W_{ia} . This switch of the “ a ” and “ b ” columns then implies that $\text{Tr}(W_{ik}^{-1}A_{kj}^T) = 0$.

3.3 An example with $B_{ij} = 0$

We conclude with an example for which $B_{ij} = 0$. Consider the following Lagrangian,

$$L = \frac{1}{4}(\dot{x}^2 + \dot{y}^2 - k)^2 - V(x, y). \quad (3.18)$$

Applying our method from sec. (3.2) we find the Hessian matrix is

$$W_{ik} = \begin{pmatrix} 3Q_x^2 + Q_y^2 - k & 2Q_x Q_y \\ 2Q_x Q_y & 3Q_y^2 + Q_x^2 - k \end{pmatrix}. \quad (3.19)$$

fixing $g_1 = Q_x$ we find that Eqs. (3.15) lead to the following equations for f_1 ,

$$\frac{\partial f_1}{\partial x} = 3Q_x^2 + Q_y^2 - k, \quad (3.20)$$

$$\frac{\partial f_1}{\partial y} = 2Q_x Q_y. \quad (3.21)$$

The solution of the previous equations is given by

$$f_1 = x(3Q_x^2 + Q_y^2 - k) + 2yQ_x Q_y. \quad (3.22)$$

Similarly we choose $g_2 = Q_y$ and solve for f_2

$$f_2 = y(3Q_x^2 + Q_y^2 - k) + 2xQ_x Q_y. \quad (3.23)$$

The inversion of $\{x, Q_x, y, Q_y\}$ in terms of $\{f_1, g_1, f_2, g_2\}$ is given by

$$x = \frac{-2f_2 g_1 g_2 + f_1(3g_1^2 + g_2^2 - k)}{9g_1^4 + 2g_1^2(g_2^2 - 3k) + (g_2^2 - k)^2}, \quad (3.24)$$

$$Q_x = g_1, \quad (3.25)$$

$$y = \frac{-2f_1 g_1 g_2 + f_2(3g_1^2 + g_2^2 - k)}{9g_1^4 + 2g_1^2(g_2^2 - 3k) + (g_2^2 - k)^2}, \quad (3.26)$$

$$Q_y = g_2, \quad (3.27)$$

and the last step is to substitute back into the canonical Hamiltonian and express it in terms of the new variables f_i and g_i ,

$$H(f, g) = \frac{3}{4}(g_1^2 + g_2^2 - \frac{1}{3}k)^2 + V[x(f_i, g_i), y(f_i, g_i)] \quad (3.28)$$

Similar to the first example, the potential $V(x, y)$ plays the role of a complicated kinetic term, while the original kinetic term becomes a Mexican hat potential.

4 CONCLUSIONS

Theories whose Lagrangian contains high powers of the velocities seem to lead to a multivalued canonical description. However, we have seen that this conclusion is not necessarily valid in many cases. Using the Dirac formalism we were able to derive a generalized definition of the canonical variables. This new definition allows additional freedom for solving the velocities in terms of the momenta. Our conclusion is that in some cases the multivalued nature of phase space originates in an inappropriate choice of coordinates rather than from a fundamental limitation. In some cases, a good choice of phase space coordinates enables one to define a single valued Hamiltonian with a standard, single valued, symplectic structure.

When a standard canonical structure can be found, we expect that the quantization can follow the standard rules, turning coordinates into operators and solving the corresponding Schrödinger equation. If one so wishes, one can transform the result back to the original coordinates by applying the formalism of canonical transformation in quantum mechanics [13]. However, one will have to address quantum ordering issues when defining the quantum Hamiltonian. Either the Poisson or Dirac brackets are noncanonical, or the Hamiltonian possesses quantum ordering ambiguities. We hope to discuss the quantization of higher-derivative theories in a future publication.

In the general case, we expect that finding good phase space coordinates is not always possible. For example, if we attempt to apply our formalism in the case of a more complicated Lagrangian, say, $L = ax^3\dot{x}^2 + b\dot{x}^3 + cx^5\dot{x}^4 - V(x)$. We will be able to find f and g satisfying Eq. (2.11). But for every solution the inversion of (x, Q) in terms of (f, g) will be multivalued. In such cases, for

which the Hamiltonian is truly multivalued, one needs a different approach. It is clear, however, that these cases are not fundamentally flawed in any way.

Even though we were able to find a Hamiltonian for the minisuperspace Gauss-Bonnet action, trying to apply the formalism for the general case will face difficulties. This is because in the case of a Lagrangian with N -degrees of freedom, the number of equations one needs to solve scales as $\binom{2N}{2}$. This technical difficulty is greatly reduced if the coupling between the coordinates and the velocities vanishes. In the general Gauss-Bonnet action, the coordinates: the metric and the velocities: the extrinsic curvature, are coupled in a nontrivial way. Progress in this direction will require the construction of an efficient method to handle the growing number of equations.

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